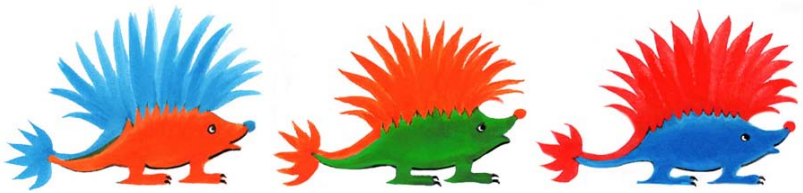
Set Theory



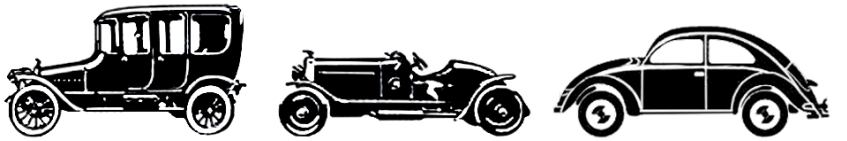
Karina Kudymova, 2013

Sets are very important in mathematics.
A set is just a collection of elements.

This is a set

$\{3, 6, 9, 12, 15\}$

This is another set



Even this is a set

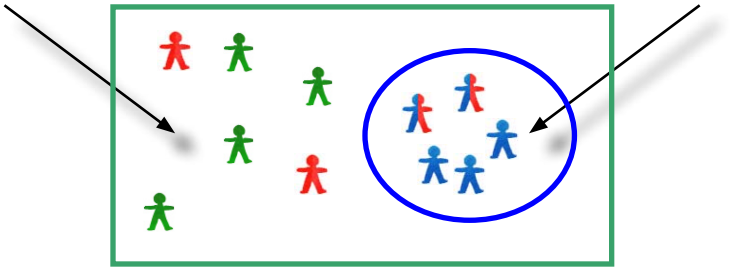
The order of elements does not matter. What matters is only which elements are included.

This is the same set, described differently.

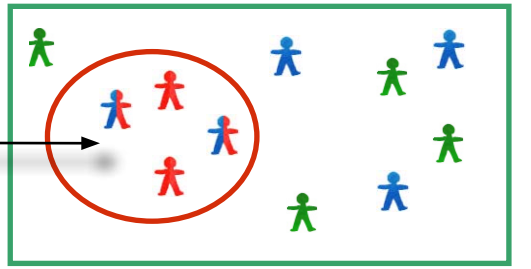
A convenient way to represent and analyze sets is to use *Venn diagrams*.

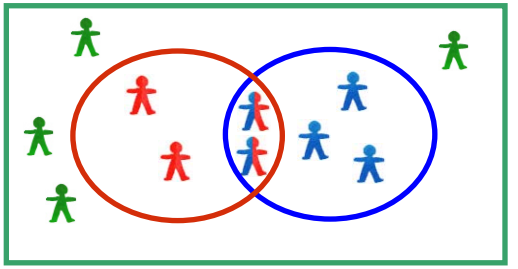
This rectangle represents all kids in a class

And the circle corresponds to kids who take the swimming class



This circle shows kids enrolled in choir





Venn diagram

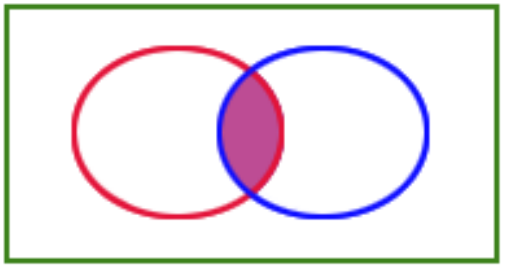
Venn diagram shows everything about the set of kids in the class: who are the siwimmers and who are the singers. And who is not singing and not swimming, and who is both swimming and singing, etc.

With Venn diagrams it is easy to show important definitions for set theory.

We'll denote sets of singers and swimmers as A and B

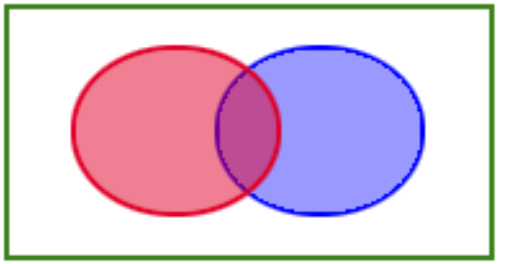
An *intersection* of two sets is also a set that contains elements present in each of two sets

$$A \cap B$$



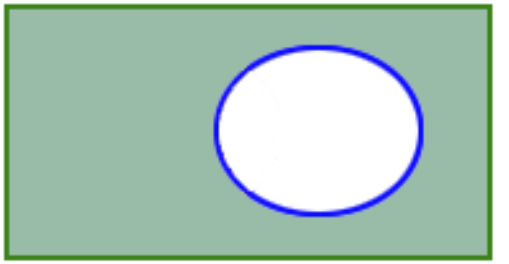
The *union* of two sets is a set with elements that either belong to one of the sets or to the other or to both

$$A \cup B$$



The *complement* of a set contain elements that don't belong to the original set

$$\overline{A}$$

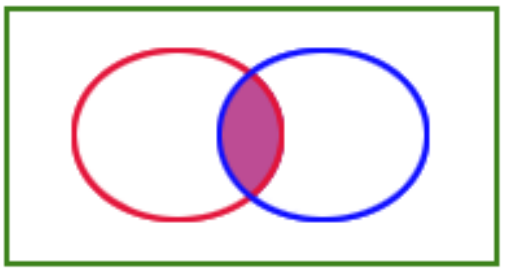


These definitions will help us prove *De Morgan's laws*.

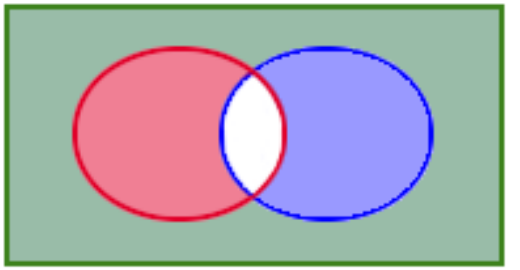
The first law says that

$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$

To prove it let's try to draw and compare the sets on both sides of this expression

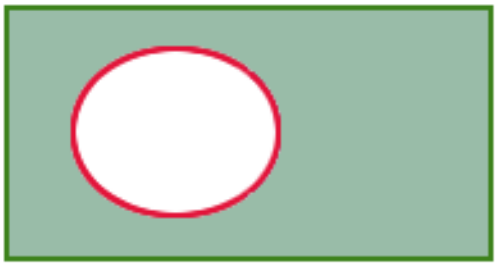


$A \cap B$

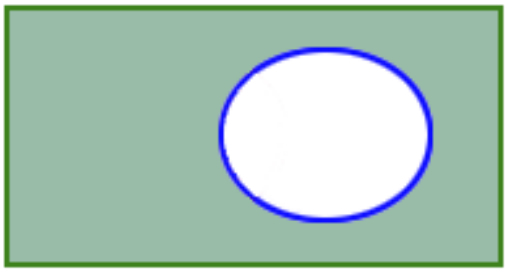


$\overline{A \cap B}$

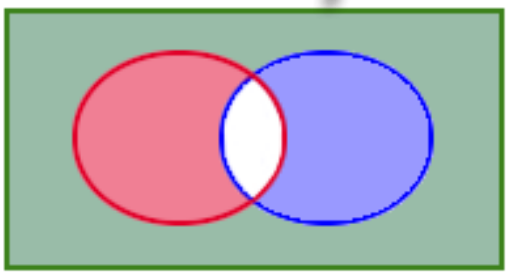
Same set



$\overline{A}$



$\overline{B}$



$\overline{A} \cup \overline{B}$

The second De Morgan law is

$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$

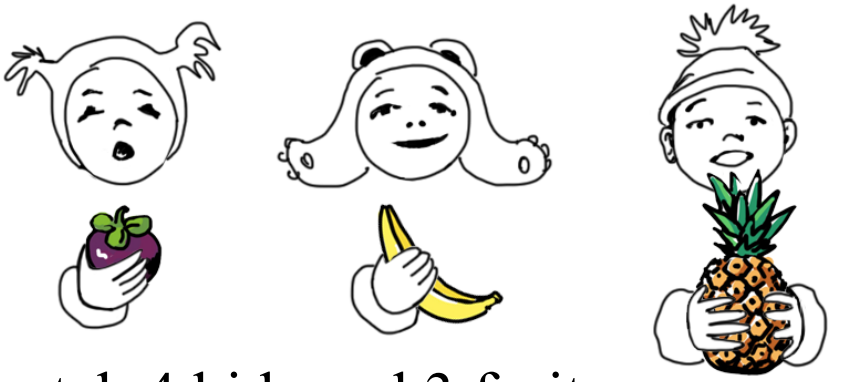
Can you prove it yourself?

What do these sets have in common?

They both have three elements. In mathematics the number of elements in a set is called *cardinality*.

Two sets have the same cardinality is if we can match each element of one set to a unique element of the other with no elements left unmatched. This is called a one-to-one correspondence, or bijection.

These sets have the same cardinality (3)



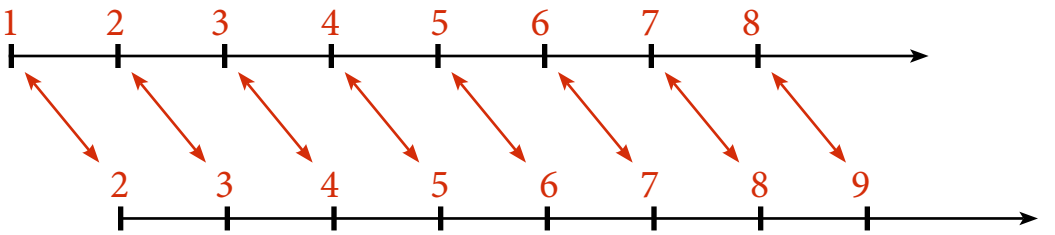
But these do not: there no way to match 4 kids and 3 fruits.

Things are pretty straightforward for finite sets, but what about infinite sets? Well, then everything is not as simple...

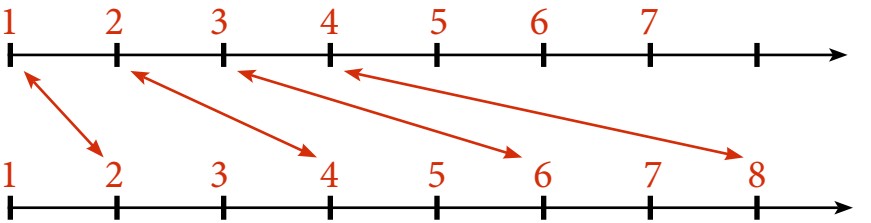
Which set is “bigger” 1,2,3,4... or 2,3,4... ?

The second set is missing element 1, so it must be “smaller”, but it turns out that these sets have the same cardinality so they are of the same size. The key is to find the right one-to-one mapping.

We can match 1 from the first set to 2 from the second, 2 from the first to 3 from the second, etc. Each element n from the first set gets paired with element n+1 from the second. That way all elements from both sets get matched, so the sets have the same cardinality.



What about the set of Natural numbers 1,2,3,4... and the set of even numbers 2,4,6,8... ? It turns out they are also of the same size. Now for each number n in the first set we pick the number 2*n in the second:

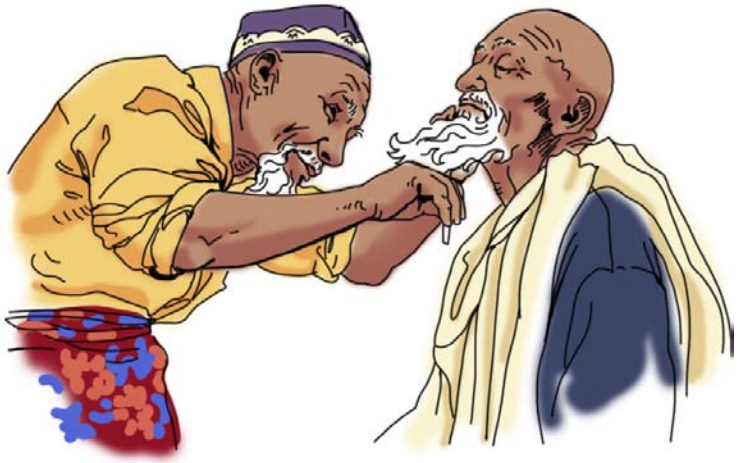


If you don't think it was weird enough, let's talk about a famous set theory paradox popularized by Bertrand Russell.
The paradox has many variations but all of the them are related to so-called *self-referencing sets*. A simple form of it is the Barber Paradox:

The town barber is a man who shaves those and only those men in town who do not shave themselves. Who shaves the barber?

If the barber shaves himself, then he is one of those men who shave themselves. But then he cannot shave himself because he shaves none of these men.

But if he doesn't shave himself, then he is one of those who don't shave themselves, and then he should shave himself.



This is one of those things in math that makes your head spin. In practice, to avoid such paradoxes, mathematicians chose not to pay much attention to self-referencing sets.