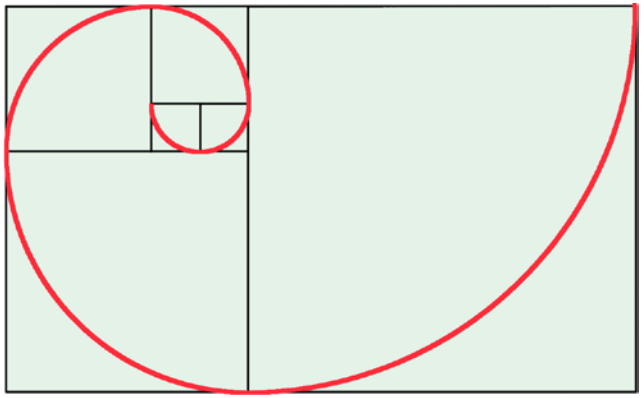
Sequences

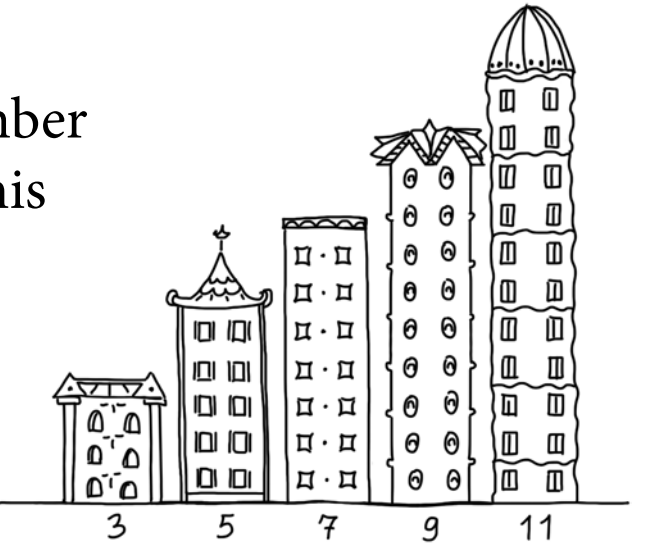


Karina Kudymova, 2013

Let's look at the sequence of Natural numbers

$$1, 2, 3, 4, 5, 6, 7, \ldots$$

How is this sequence formed? Each number in equal to the previous one plus one. This is an example of an *arithmetic sequence*.



An arithmetic sequence or arithmetic progression is a sequence of numbers such that the difference between any number and the previous one is constant.

This is an arithmetic sequence.

$$3,\ 7,\ 11,\ 15,\ 19$$

$$4 \quad 4 \quad 4 \quad 4$$

But this is not

$$5,\ 7,\ 9,\ 10,\ 12$$

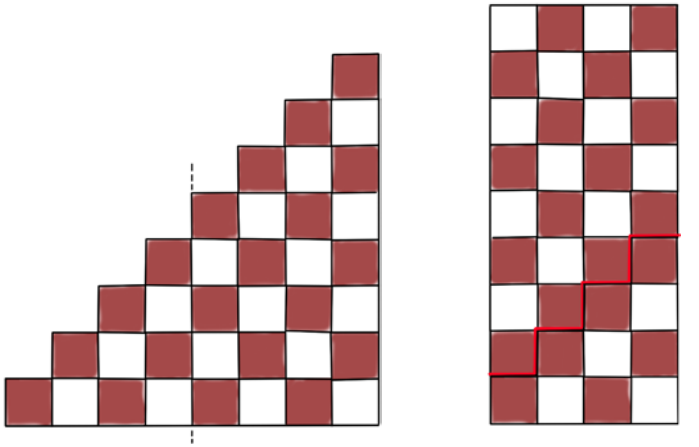
$$2 \quad 2 \quad 1 \quad 2$$

In mathematics, sequences never end. But if we truncate an arithmetic progression and take the sum of corresponding elements that's called *arithmetic series*.

Let's compute 1+2+3+4+5+6+7+8

$$1+2+3+4+5+6+7+8=9\times4=36$$

We can pair the first element with the last, second with second to last, and so on. Here we have four pairs each adding up to 9, so the total sum is 36.



This trick works for larger numbers as well:

$$1+2+3+...+998+999+1000= 10\times01\ 500=500500$$

In general, to find the sum of elements of arithmetic progression, add the first and the last values and multiply by the number of elements you are adding and divide by two.

Can you find the sum:

$$3+7+11+15+19$$

If the number of elements is odd, the middle value is not paired, but the formula still works.

Arithmetic sequences are formed by adding a constant value to the previous one, but what if we use multiplication instead of addition? Then we get a *geometric sequence*.

Geometric sequence or geometric progression is a sequence of numbers in which each element formed by the previous one multiplied by a fixed number

$$1, 2, 4, 8, 16, 32, 64 \ldots$$

Let's derive a formula to add n elements of this sequence:

$$1+2+2^2+ \ldots 2^{n-2}+2^{n-1} = x$$

If we multiply both sides of this equality by 2 we get

$$2+2^2+ \ldots 2^{n-1}+2^n = 2x$$

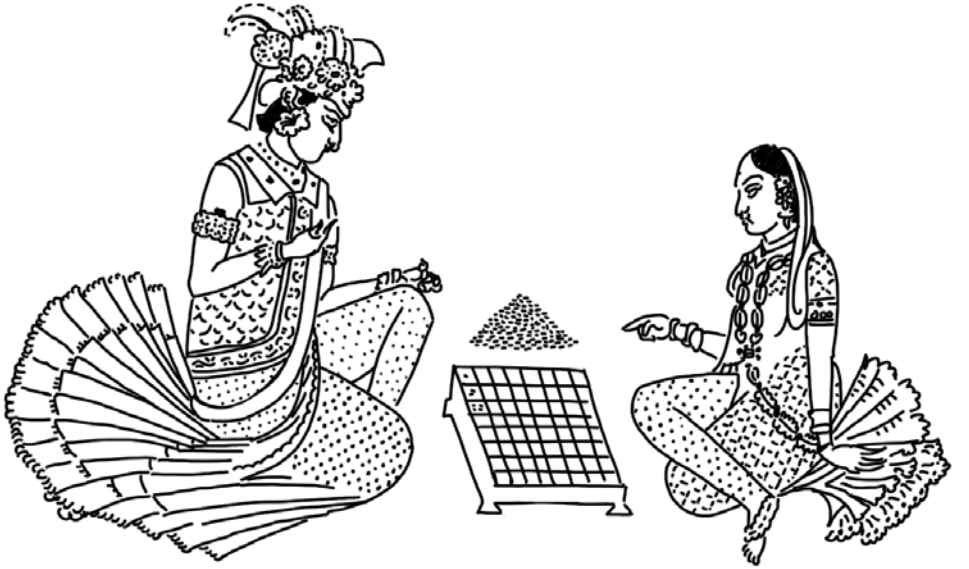
But on the left we have the same sequence without the first term 1 with additional 2^n:

$$x-1+2^n=2x$$

Then $x=2^n-1$ or $1+2+2^2+ \ldots+2^{n-1}=2^n-1$

Can you check it with $n = 5$?

There is a legend that the inventor of the game of chess showed it to the ruler of his country who was so impressed that offered the inventor any reward he wished. The inventor asked the ruler to put a grain of rice on the first square of the chess board, two on the second, four on the third, and keep doubling the number of grains until all 64 squares are covered. Not realizing how big this reward would be, the ruler accepted the proposition. How much rice did he offer?



We know that $1+2+4+8+\ldots\ 2^{63}=2^{64}-1$

And $2^{64}-1=18{,}446{,}744{,}073{,}709{,}551{,}615$

That's a very large number. A heap of rice with that many grains would be taller than the tallest mountain on Earth.

In geometric progressions numbers don't have to increase, they can decrease too. This is a geometric progression:

$$\tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{8}, \tfrac{1}{16}, \tfrac{1}{32}, \ldots$$

Can we use the same trick to find the sum of elements of this progression?

$$\tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} + \tfrac{1}{32} + \ldots = x$$

Note that this time we don't truncate the sequence, but are adding infinitely many terms! Have no fear, we can do it!

$$1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} + \tfrac{1}{32} + \ldots = 2x$$

Let's multiply both sides by 2. On the left we get the original sequence (x) with additional 1, so the left hand side becomes x+1.

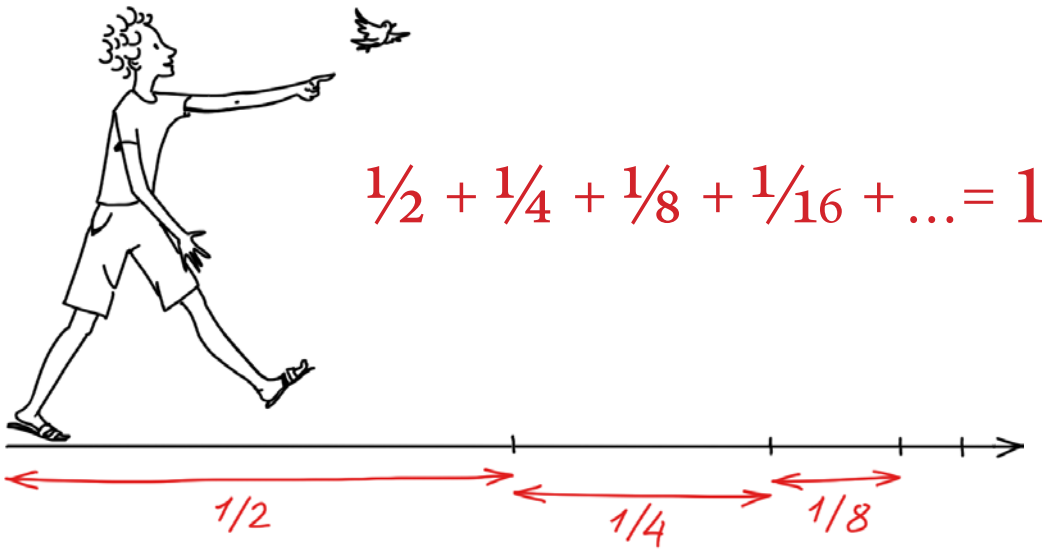
Then we get $x + 1 = 2x$ $\qquad x = 1$

or $$\tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} + \tfrac{1}{32} + \ldots = 1$$

This result is related to the ancient paradox formulated by the Greek philosopher Zeno of Elea

If a traveller sets on a journey from one city to another first he'll have to travel half of the required distance, the half of the remaining part, or one quarter of the full distance, then half of the remaining part, and so on. How can the traveller ever reach the destination if he has to travel infinite number of distances to get there?

But we already know that even though there are infinitely many terms in this sequence



$$\tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} + \ldots = 1$$

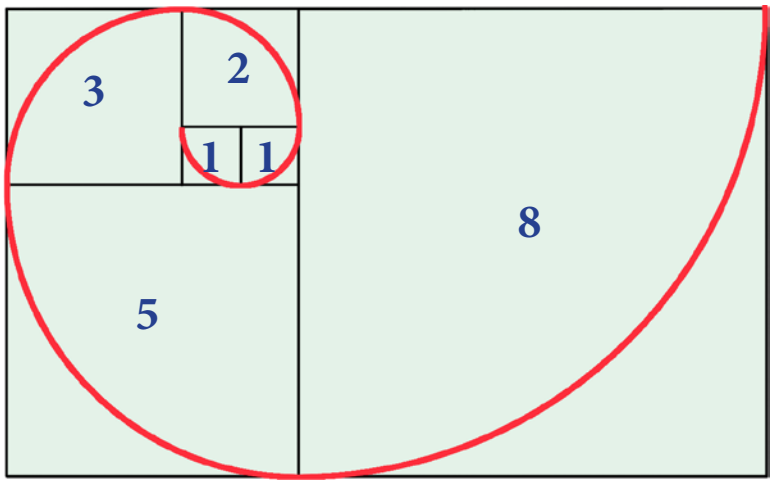
their sum is finite and is equal to 1.

In addition to arithmetic and geometric progressions, there are other well-studied sequences in Mathematics. One example is the Fibonacci sequence.

The first two elements of the sequence are 1s and then each element is equal to the sum of the two that come before it:

1, 1, 2, 3, 5, 8, 13, 21, 34, 55...

Fibonacci numbers have many interesting properties. For example, they can be used to construct the Fibonacci spiral.



Can you see how Fibonacci numbers are used to construct this spiral?

Also, as Fibonacci numbers get bigger, the ratio of two consecutive numbers becomes closer and closer to the golden ratio often used in design and architecture.