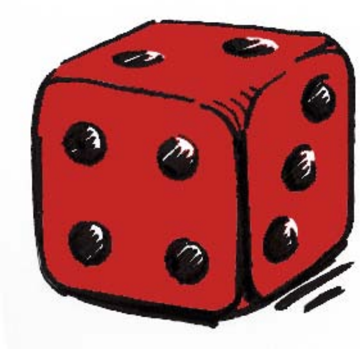
Probability



Do you know that if we toss a coin the chances of heads and tails are *50:50*? Do you know what it means? It just means that on average *50%* of the tosses (half the time) will end up in heads and *50%* (the other half) in tails.

50% in this example is called *probability*. In general, if an experiment that can produce a number of outcomes we define probability of event A as

$$P(A) = \frac{\textit{Number of outcomes producing A}}{\textit{Total number of outcomes}}$$

For a coin, there are two outcomes: heads and tails

$$P(heads) = P(tails) = 1/2 = 50\%$$

Now, suppose that you tossed a coin 5 times and every time it showed heads. Is the next time you try more likely to end up with tails to compensate all the heads in previous experiments?

No. The coin has no memory. Every time you flip it, the probability of heads and tails is the same: *50%*. What is true, though, is the *law of large numbers*: as we keep tossing a coin for a long time and keep track of the results, the number of heads and tails will be getting closer to *50%*.

What does *20%* chance of rain mean?

The law of large numbers also explains why all the molecules in a room don't gather in a single corner, but are spread out evenly.

Instead of a coin, let's look at a dice. If we roll it, what is the probability of the dice showing a **3**?

There are 6 possible outcomes and **3** is one of them, so the probability is *1/6*.

And the probability of a dice showing **3** or a higher number? That will happen with either of four outcomes **3**, **4**, **5**, and **6**, so the probability is *4/6=2/3*.

Now, what's the probability of a dice showing a number less than **3**? Two outcomes will work here: **1** and **2**, so the probability is *2/6=1/3*.

This illustrates the *subtraction rule of probability*:

The probability that event **A** will occur is equal to *1* minus the probability that event **A** will not occur.

Here, event **A** is that the dice shows number **3** or greater and **A** not occurring is the dice showing less than **3**.

*P(***A** not occurring*)=1-P(***A***)=1-4/6=2/6*.

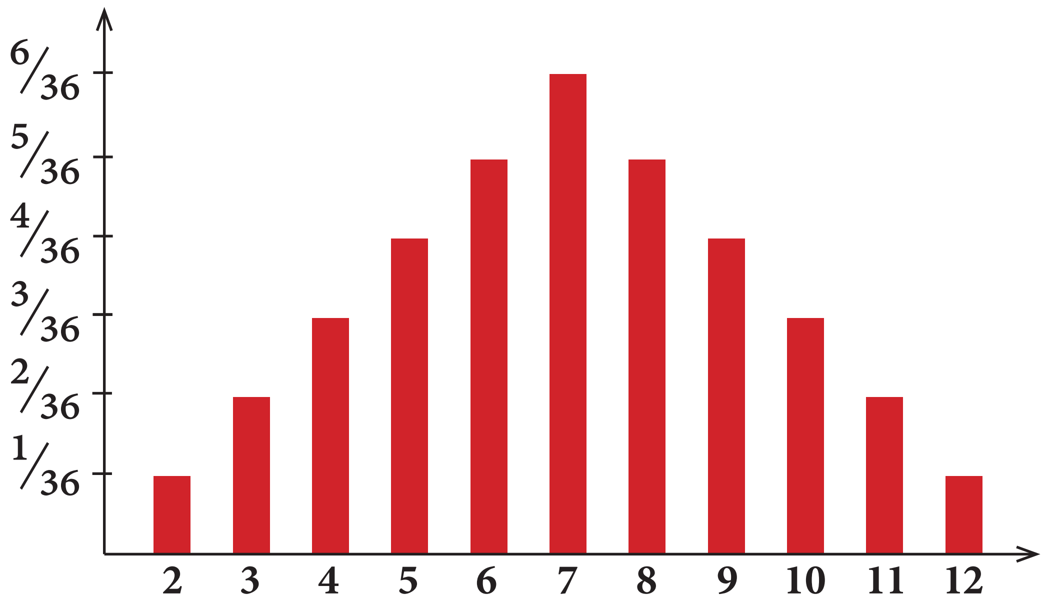
If we are throwing two dice, what combined value has higher probability, **5** or **10**?

If one dice is red and the other is green, there are 36 total combinations, out of which there are four ways to get a **5** and three ways to get a **10**:

5: (1+4) (2+3) (3+2) (4+1)
10: (4+6) (5+5) (6+4)

The probability of getting a **5** is *4/36=1/9* and of getting a **10** is *3/36=1/12*

2: (1,1)
3: (1,2)(2,1)
4: (1,3)(2,2)(3,1)
5: (1,4)(2,3)(3,2)(4,1)
6: (1,5)(2,4)(3,3)(4,2)(5,1)
7: (1,6)(2,5)(3,4)(4,3)(5,2)(6,1)
8: (2,6)(3,5)(4,4)(5,3)(6,2)
9: (3,6)(4,5)(5,4)(6,3)
10: (4,6)(5,5)(6,4)
11: (5,6)(6,5)
12: (6,6)



Note that the sum probabilities of all 12 cases is *1*, as it should be

Probability theory has also the *rule of addition*: The probability that Event **A** or Event **B** occurs is equal to the probability that Event **A** occurs plus the probability that Event **B** occurs minus the probability that both Events **A** and **B** occur.

$$P(\mathbf{A} \cup \mathbf{B}) = P(\mathbf{A}) + P(\mathbf{B}) - P(\mathbf{A} \cap \mathbf{B})$$

For example, the probability that a dice will show a number that is either even or divisible by 3 is equal to the probability that a number is even (*1/2*) plus the probability that this number is divisible by three (*1/3*) minus the probability that it's both divisible by two and three (*1/6*). So it is *1/2+1/3-1/6=4/6*.

There is also the *rule of multiplication*: The probability that Events **A** and **B** both occur is equal to the probability that event **A** occurs, times the probability that event **B** occurs, given that **A** has occurred.

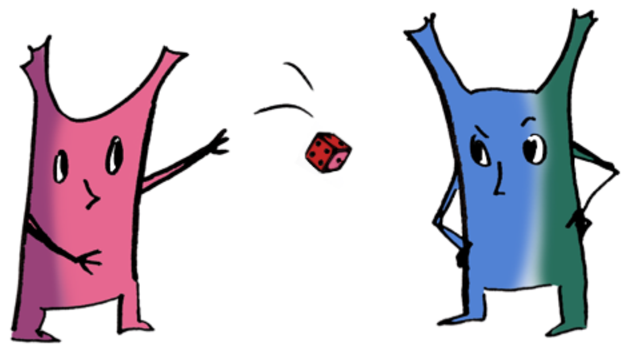
$$\mathrm{P}(\mathbf{A} \cap \mathbf{B}) = \mathrm{P}(\mathbf{A})\ \mathrm{P}(\mathbf{B} | \mathbf{A})$$

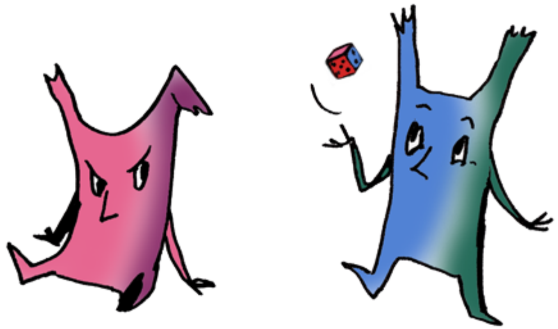
The probability that a number is divisible by six is the probability that it is even (*1/2*), times the probability that it's divisible by three given that it's even (*1/3*), or *1/2*1/3= 1/6*.

To put it all together, let's solve a puzzle...

> Two players are alternating throwing a dice. The first one to get a **6** wins. What's the probability for the player who throws first to win?

Clearly, the player who starts the game has some advantage, but what is it? Suppose p is the probability for the first player to win, then (subtraction rule) the probability for the first player to lose is $1\text{-}p$.





If the first player gets a **6** on the first move (probability *1/6*), she wins. But she can also win later even if she didn't win right away. In this case, the second player effectively starts a new game and has probability *1- p* to lose. First player's probability to win (addition rule) is the probability to win with the first throw (*1/6*) plus the probability to get to the second move (*5/6*) times the probability for the second player to lose.

$$p = 1/6 + 5/6\ (1-p)$$
$$6p = 1 + 5*(1-p)$$
$$11p = 6$$
$$p = 6/11$$

So the probability for the first player to win is *6/11* and for the second player to win is *1- 6/11 = 5/11*