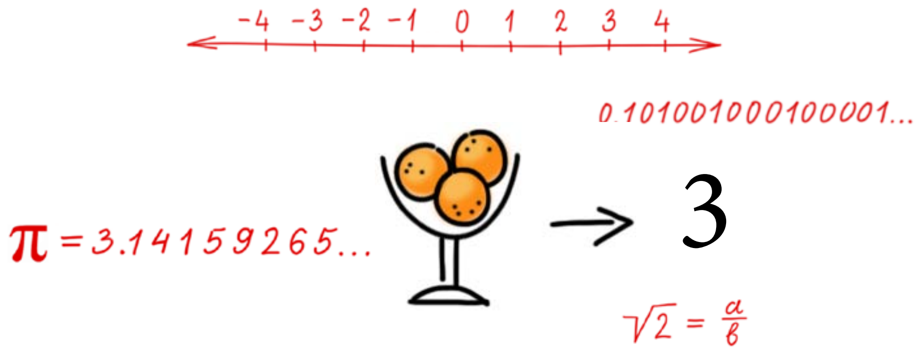
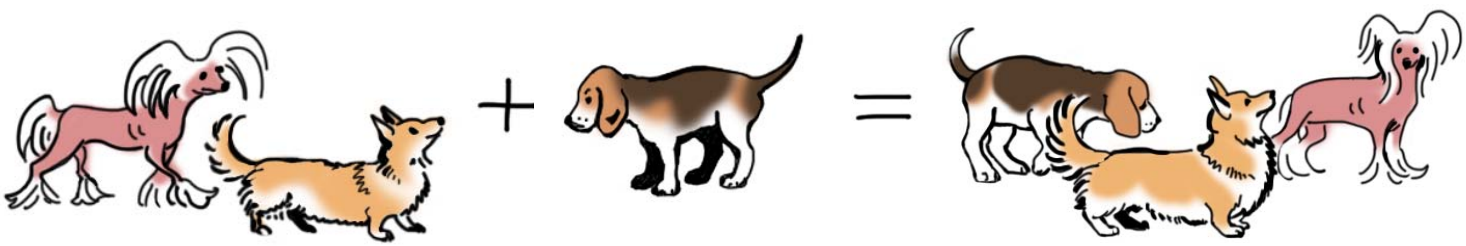
Types of Numbers



The most basic numbers are used for counting things: one, two, three ...

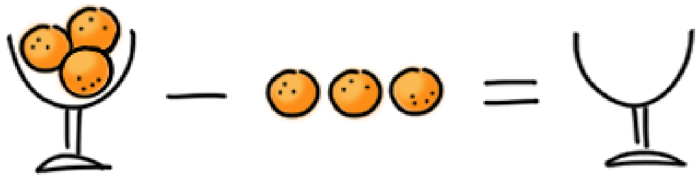
They are so fundamental in mathematics that they are called *natural numbers*.

Natural numbers serve us well in simple calculations, including additions and subtractions.



But soon we realize that just having natural numbers is not enough for counting.

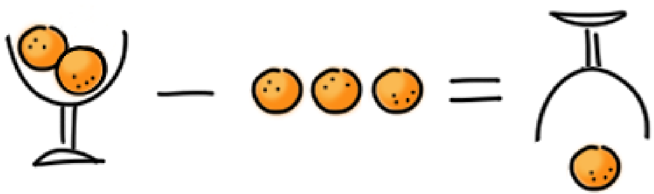
If we start with three oranges and remove three of them, how many are left?



0

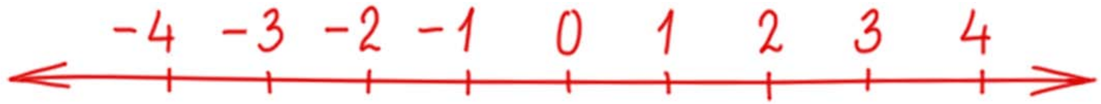
Inventing *zero* was a major breakthrough in ancient times, but it didn't end there.

If I have two oranges but I am supposed to give you three, how many will I have left? Well, I won't have any oranges, besides I'll have figure out where to get an orange to pay back. That's where *negative* numbers come to help...



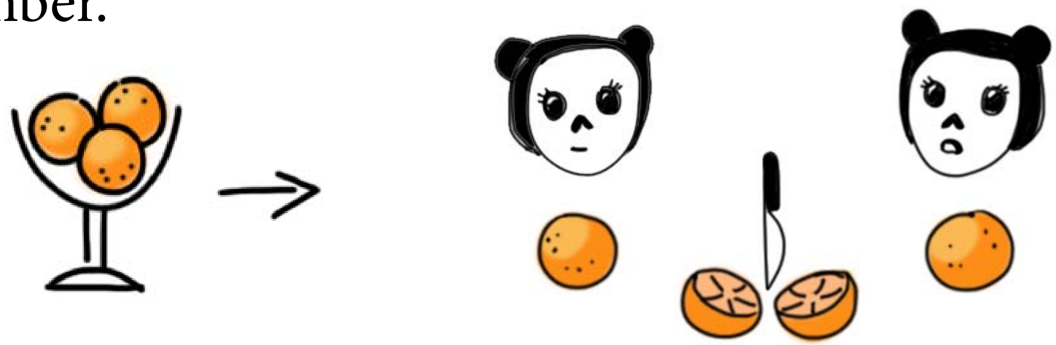
-1

Natural numbers (1, 2, 3 ...), zero and negative numbers (-1, -2, -3) combined are called *integer numbers.*



We can take any two integer numbers and add, subtract or multiply them and the result will always be an integer number.

But we have a problem with division. If we divide three oranges equally between two people each gets one and a half, and that's not an integer number.



Rational numbers come to the rescue. They are of the form a/b where both a and b are integers.

$\frac{3}{5}$ $\frac{27}{119}$ $\frac{1}{3}$ $\frac{13597}{457129}$

With rational numbers, we can add, subtract, multiply and divide them, and still get a rational number as a result (as long as we don't divide by zero).

Of course rational numbers include integer numbers because they can be written as the ration of themselves and 1

$$5 = \frac{5}{1} \qquad 28 = \frac{28}{1}$$

Rational numbers can be represented in a decimal form. There is a theorem that rational numbers in decimal form either terminate or have repeated patterns.

$$\frac{1}{4} = 0.25 \qquad \frac{2}{3} = 0.666\ldots = 0.\overline{6} \qquad \frac{12}{7} = 1.\overline{714285}$$

Can you think of a decimal fraction that is infinite and doesn't repeat itself? How about $0.101001000100001\ldots$

Here 1s are interrupted by increasing numbers of 0s. Can you tell why it's not a repeating decimal?

And because it's not, it cannot be a rational number.

Numbers that are not rational are called *irrational*. A well known irrational number is square root of two. The proof of it goes back to the Ancient Greeks.
Let's assume that $\sqrt{2}$ is rational, then we could write it as a fraction

$$\sqrt{2} = \frac{a}{b}$$

Any fraction can be brought to an irreducible form where numerator and denominator don't have common divisors greater than 1. We'll assume that our fraction is already in this form.

$$\sqrt{2} = \frac{a}{b} \Rightarrow 2 = \frac{a^2}{b^2} \Rightarrow a^2 = 2b^2$$

We won't prove this theorem here, hope you can take our word for it for now.

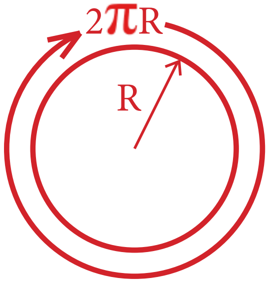
Then a must be divisible by 2 (can you tell why?).

Then it can be written as $a = 2c$

$$(2c)^2 = 2b^2 \Rightarrow 4c^2 = 2b^2 \Rightarrow b^2 = 2c^2$$

But then b must be divisible by 2, then the fraction is reducible. We get a contradiction, which proves that our initial hypothesis that $\sqrt{2}$ is rational is false.

Another well-known irrational number is π (pi) — the ratio of a circle's circumference to its diameter.



It's irrational so its decimal form has infinitely many digits that never enter a repeating pattern.

3.14159265358979323846264338327950288419716939937510582097494459230781640628620899862803482
53421170679821480865132823066470938446095505822317253594081284811174502841027019385211055596
44622948954930381964428810975665933446128475648233...

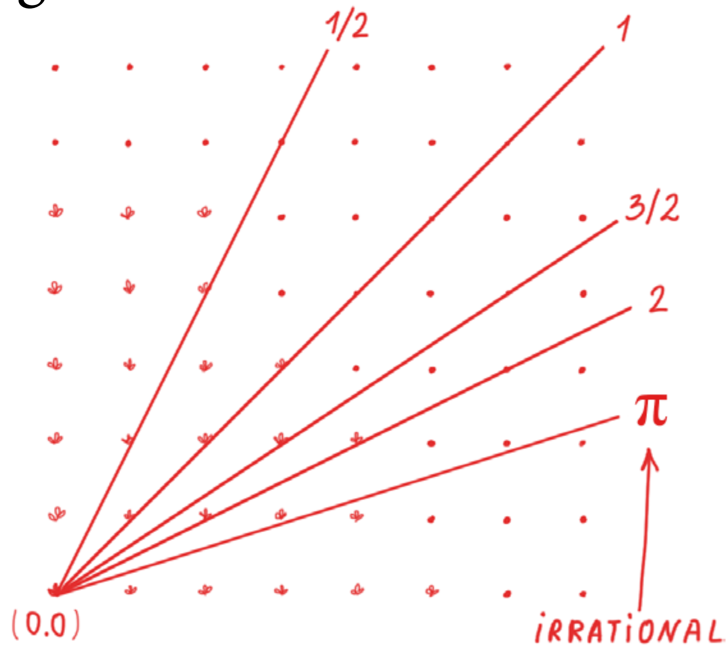
How many digits can you memorize? You can use this sentence in which the numbers of letters in each word correspond to π digits

How I wish I could recollect pi easily today!

π = 3. 1 4 1 5 9 2 6 5 ...

π is not only irrational, it is also a transcendental number. In a transcendental number every sequence of digits you can think of occurs in it infinitely many times.

So there are rational numbers, which are representable as ratios of integers, and irrational numbers, that are not. Combined they form *real* numbers.
There are infinitely many rational numbers and irrational numbers, but in some sense, there are much more irrationals than rationals, which is an amazing fact.



Think of an "integer lattice", dots with integer coordinates on a plane (like grapes in a vineyard). For every rational number a/b there is a ray that starts from the origin (0,0) and passes through the point (a,b).

Rays corresponding to irrational numbers will miss *all* the lattice points. There is a theorem that probably of a random ray hitting a lattice point is zero.

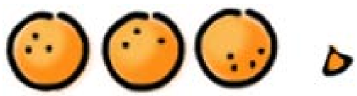
In other words, if we look from the origin in a random direction, most likely we won't see a single point!

Are real numbers sufficient for everything? No, real numbers don't offer a solution to equation $x^2+1=0$

Mathematicians came up with yet another class of numbers: complex numbers. They combine real numbers with imaginary ones and are based on number i with the property

$$i^2=-1$$

Imaginary numbers are rather freaky. You can at least approximate π oranges...



but you cannot even imagine i oranges.

$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Mathematicians are very inventive, they didn't stop there. They came up with other objects that they add or multiply. They include vectors, matrices and tensors.