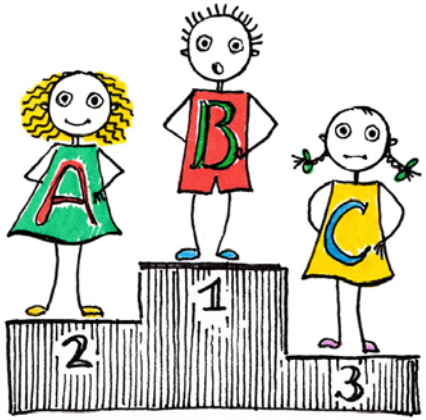
Combinatorics



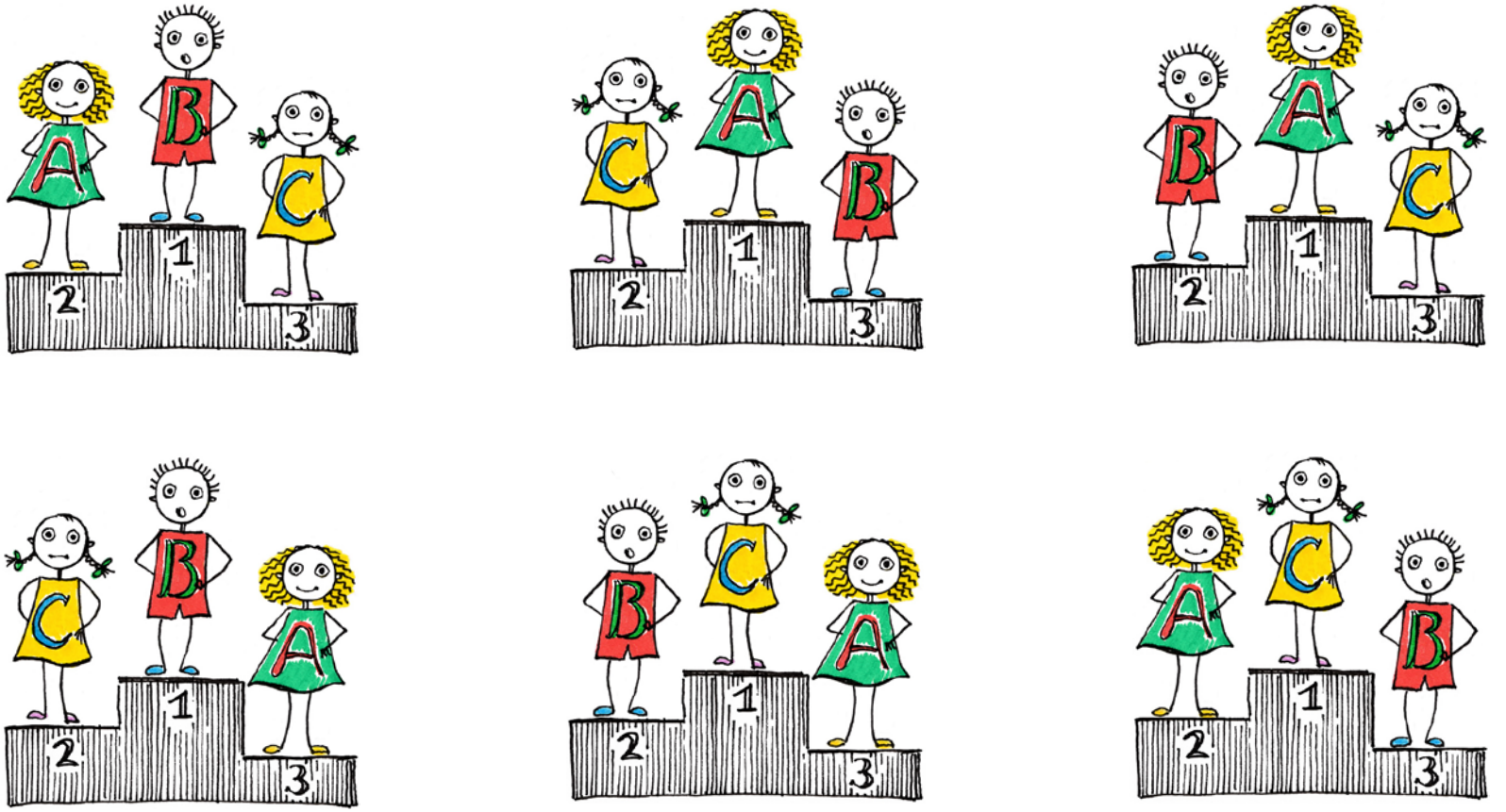
Karina Kudymova, 2013

Three classmates, Alice, Brian, and Carol run a mile.

One of them wins, another comes second, and the other comes last.

How many different combinations are possible?

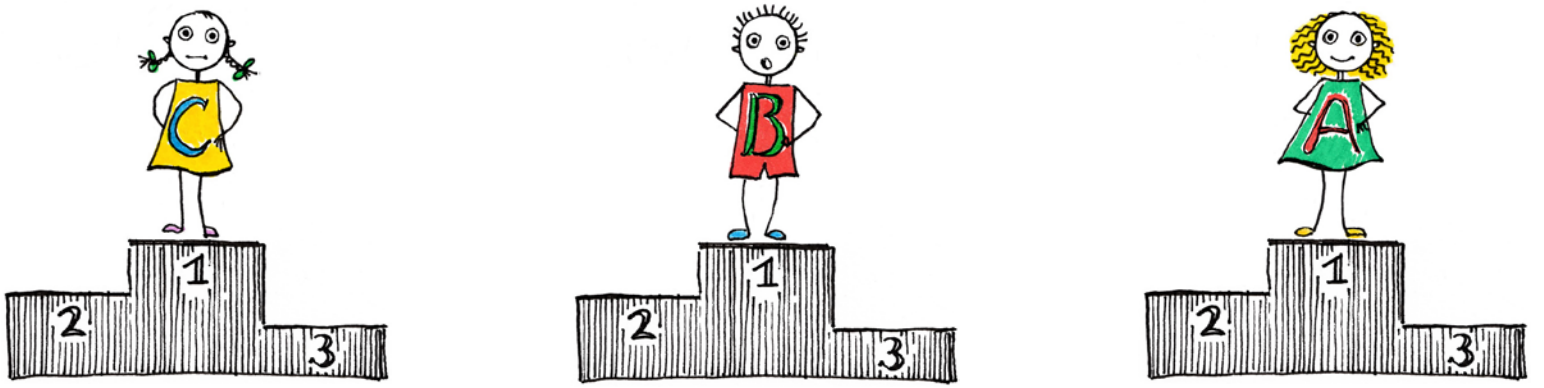
Let's see... We can list them all:



So far so good, but what if we had 10 kids and ten places? we'd have too many combinations to list

Can we count combinations for Alice, Brian and Carol differently?

The winner can be either of the three kids



If Alice is a winner the second place
can go to either Brian or Carol

And whoever is left gets
the third place

First	Second	Third	Result
Alice	Brian	Carol	1- Alice, 2 - Brian, 3 - Carol
	Carol	Brian	1- Alice, 2 - Carol, 3 - Brian
Brian	Alice	Carol	1- Brian, 2 - Alice, 3 - Carol
	Carol	Alice	1- Brian, 2 - Carol, 3 - Alice
Carol	Alice	Brian	1- Carol, 2 - Alice, 3 - Brian
	Brian	Alice	1- Carol, 2 - Brian, 3 - Alice

Number of possibilities			Total
3	2	1	$3\times2\times1 = 6$

In other words: each of the three kids could come first, any of the remaining two could come second, and the one who is left comes third. The number of combinations is $3\times2\times1=6$.

A note on factorials

An expression like $1\times2\times3$, which is a product of all positive number less than or equal to a given one, is called factorial and is written as $3!$

$$3!=1\times2\times3=6$$

Factorials are so useful in various areas of mathematics that most scientific calculator have a special key for them.

$$5!=1\times2\times3\times4\times5=120$$

Factorials can get big quickly. Try to find 10 factorial. What about 20 factorial?

What if there were ten kids competing for ten places?

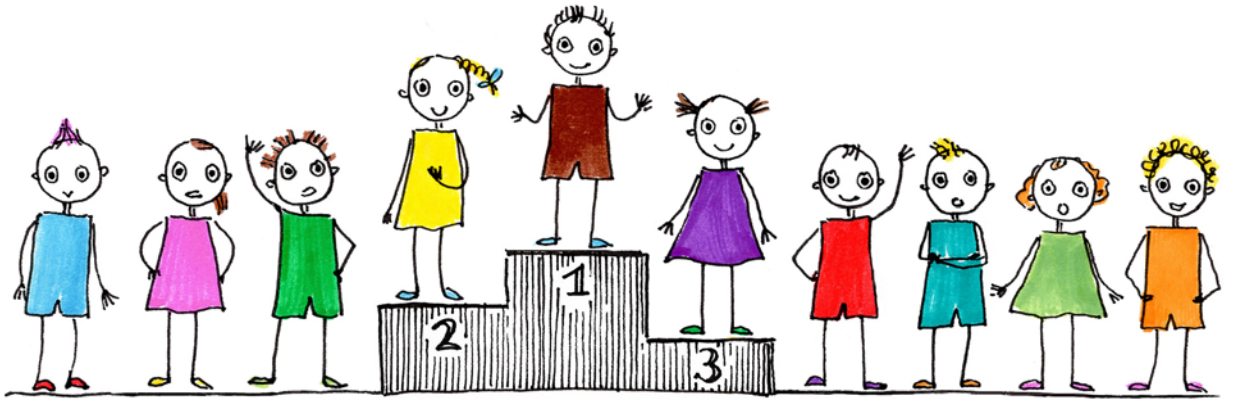
The solution would be similar: either of the ten can be first, any of the remaining nine can be second, etc.
The total number is $10\times9\times8\times7\times6\times5\times4\times3\times2\times1$ or $10!$

This is the number of permutations of 10 elements. There are $10!$ ways to arrange 10 of anything in 10 places.

Can you calculate 10! ?
It's 3628800.

Why does 10! end with two zeros?
How many zeros are there at the end of 20! ? What about 30! ?

What if ten kids compete only
for three medals?



Again, any of ten can get a gold, any of the remaining 9 can get a silver and any of remaining 8 can get a bronze. The answer is $10\times9\times8$ and is called *3-permutations of 10*.

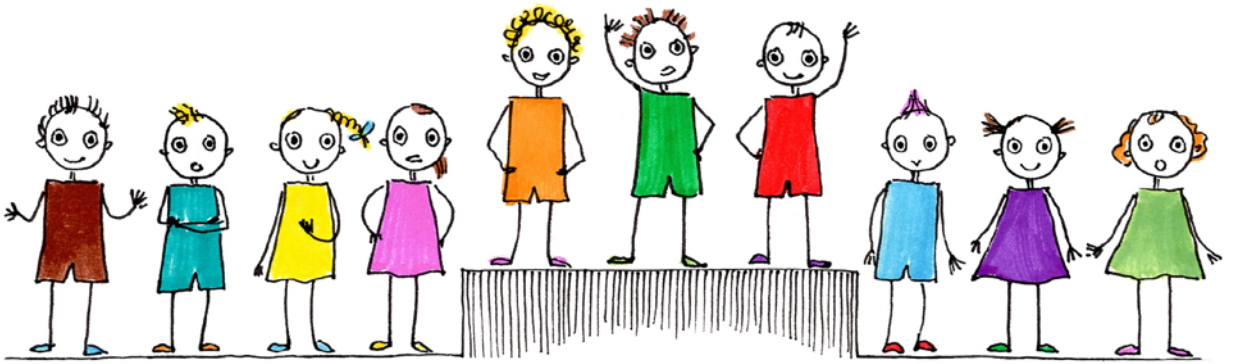
Can you calculate it?

Can we write it using factorials?

Sure:

$10\times9\times8 = 10! / 7! = 10! / (10-3)!$

In the final problem, let's assume that top three runners out of ten get to represent their school in a district competition. How many choices are there?



This problem is very similar to the last one with one exception: the order of the first three does not matter.

The solution was $10\times9\times8$ permutations, but each triplet was counted multiple times corresponding to different places.

We already know that there are 6 different ways for top three runners to with the first three places. In $10\times9\times8$ we counted each solution 6 or 3! times.

The new solution is

$10\times9\times8 / 3! = 10! / (7! \times 3!) = 10! / ((10-3)! \times 3!)$

This expression is called 3-*combinations* of 10.